National Committee for Mathematical Contests

Further International Selection Test

May 8, 1980 $3\frac{1}{2}$ hours

1. VLMN and VABC are tetrahedra with A, B, C on VL, VM, VN, produced as necessary. The in-centre of triangle LMN coincides with the centroid of triangle ABC.
   (i) Determine VA, VB, VC in terms of the sides of triangle LMN and VL, VM, VN.
   (ii) Determine the condition that the tetrahedra have equal volumes.
   (iii) If the tetrahedra have unequal volumes, determine, with proof, which has the greater volume.

2. Determine, with proof, all the prime numbers in the sequence (u_n) of integers defined by

$$u_0 = 2, \quad u_1 = 3,$$

$$u_{n+2} = u_{n+1}u_n - u_{n+1} - u_n + 2 \qquad (n \geqslant 0).$$

3. Prove that if $a_0 = 0, a_1, a_2, \ldots\ldots, a_n$ are real numbers, then

$$\sum_{i=1}^{n} a_i(a_i - a_{i-1}) \leqslant \tfrac{1}{2}(n+1)\sum_{i=1}^{n}(a_i - a_{i-1})^2,$$

equality holding if and only if $a_i = ia_1$ $(0 \leqslant i \leqslant n)$.

4. Given a set of n people, it is desired to arrange a series of bridge games such that every two of the n people play as opponents in exactly one game.
   Show that this can be done if and only if n is of the form $n = 8m+1$, where m is a positive integer.
   (There is no restriction on the number of times, if any, two people play as partners.)